IV. *A most compendious and facile Method for Constructing the Logarithms, exemplified and demonstrated from the Nature of Numbers, without any regard to the Hyperbola, with a speedy Method for finding the Number from the Logarithm given.* *By* E. Halley.

THE Invention of the Logarithms is justly esteemed one of the most Useful Discoveries in the Art of Numbers, and accordingly has had an Universal Reception and Applause; and the great Geometricians of this Age have not been wanting to cultivate this Subject with all the Accuracy and Subtilty a matter of that consequence doth require; and they have demonstrated several very admirable Properties of these Artificial Numbers, which have rendred their Construction much more facile than by those operose Methods at first used by their truly Noble Inventer the Lord *Napeir*, and our worthy Country-man Mr. *Briggs*.

But notwithstanding all their Endeavours, I find very few of those who make constant use of Logarithms, to have attained an adequate Notion of them, to know how to make or examine them; or to understand the extent of the use of them: Contenting themselves with the Tables of them as they find them, without daring to question them, or caring to know how to rectifie them, should they be found amiss, being I suppose under the apprehension of some great difficulty therein. For the sake of such the following Tract is principally intended, but not without hopes however to produce something that may be acceptable to the most knowing in these matters.

But first, it may be requisite to premise a definition of Logarithms, in order to render the ensuing Discourse more clear, the rather because the old one *Numerorum proportionalium æqui differentes comites*, seems too scanty to define them fully. They may much more properly be said to be *Numeri Rationum Exponentes:* Wherein we consider *ratio* as a *Quantitas sui generis*, beginning from the *ratio* of *equality*, or 1 to 1 = 0; being

Affir-

Affirmative when the *ratio* is increaſing, as of Unity to a greater Number, but Negative when decreaſing; and theſe *rationes* we ſuppoſe to be meaſured by the Number of *ratiunculæ* contained in each. Now theſe *ratiunculæ* are ſo to be underſtood as in a continued Scale of Proportionals infinite in Number between the two terms of the *ratio*, which infinite Number of mean Proportionals is to that infinite Number of the like and equal *ratiunculæ* between any other two terms, as the Logarithm of the one *ratio* is to the Logarithm of the other. Thus if there be ſuppoſed between 1 and 10 an infinite Scale of mean Proportionals, whoſe Number is 100000 *&c. in infinitum*; between 1 and 2 there ſhall be 30102 *&c.* of ſuch Proportionals, and between 1 and 3 there will be 47712 *&c.* of them, which Numbers therefore are the Logarithms of the *rationes* of 1 to 10, 1 to 2, and 1 to 3; and not ſo properly to be called the Logarithms of 10, 2 and 3.

This being laid down, it is obvious that if between Unity and any Number propoſed, there be taken any infinity of mean Proportionals, the infinitely little augment or decrement of the firſt of thoſe means from Unity, will be a *ratiuncula*, that is, the *momentum* or *Fluxion* of the *ratio* of Unity to the ſaid Number: And ſeeing that in theſe continual Proportionals all the *ratiunculæ* are equal, their Sum, or the whole *ratio* will be as the ſaid *momentum* is directly; that is, the Logarithm of each *ratio* will be as the Fluxion thereof. Wherefore if the Root of any Infinite Power be extracted out of any Number, the *differentiola* of the ſaid Root from Unity, ſhall be as the Logarithm of that Number. So that Logarithms thus produced may be of as many forms as you pleaſe to aſſume infinite *Indices* of the Power whoſe Root you ſeek: as if the *Index* be ſuppoſed 100000 *&c.* infinitely, the Roots ſhall be the Logarithms invented by the Lord *Napeir*; but if the ſaid *Index* were 2302585 *&c.* Mr. *Briggs*'s Logarithms would immediately be produced. And if you pleaſe to ſtop at any number of Figures, and not to continue them on, it will ſuffice to aſſume an *Index* of a Figure or two more than your intended Logarithm is to have, as Mr. *Briggs* did, who to have his Logarithms true to 14 places, by continual extraction of the Square Root, at laſt came to have the Root of the 140737488355328 *th.* Power; but how operoſe that Extraction was, will be eaſily judged by whoſo ſhall undertake to examine his *Calculus*.

Now

Now, though the Notion of an Infinite Power may seem very strange, and to those that know the difficulty of the Extraction of the Roots of High Powers, perhaps impracticable; yet by the help of that admirable Invention of Mr. *Newton*, whereby he determines the *Unciæ* or Numbers prefixt to the Members composing Powers (on which chiefly depends the Doctrine of Series) the Infinity of the Index contributes to render the Expression much more easie: For if the Infinite Power to be resolved be put (after Mr. *Newton*'s Method) $\overline{p+pq}$, $\overline{p+pq}|^{\frac{1}{m}}$ or $\overline{1+q}|^{\frac{1}{m}}$, instead of $1+\frac{1}{m}q + \frac{1-m}{2mm}qq + \frac{1-3m+2mm}{6m^3}q^3 + \frac{1-6m+11mm-6m^3}{24m^4}q^4$ &c. (which is the Root when m is finite,) becomes $1+\frac{1}{m}q - \frac{1}{2m}qq + \frac{1}{3m}q^3 + \frac{1}{4m}q^4 + \frac{1}{5m}q^5$, &c. mm being *infinitè* infinite, and consequently whatever is divided thereby vanishing. Hence it follows that $\frac{1}{m}$ multiplied into $q - \frac{1}{2}qq + \frac{1}{3}qqq - \frac{1}{4}q^4 + \frac{1}{5}q^5$ &c. is the augment of the first of our mean Proportionals between Unity and $1+q$, and is therefore the Logarithm of the *ratio* of 1 to $1+q$; and whereas the Infinite Index m may be taken at pleasure, the several Scales of Logarithms to such *Indices* will be as $\frac{1}{m}$ or reciprocally as the *Indices*. And if the Index be taken 10000 &c. as in the case of *Napeir*'s Logarithms, they will be simply $q - \frac{1}{2}qq + \frac{1}{3}qqq - \frac{1}{4}q^4 + \frac{1}{5}q^5 - \frac{1}{6}q^6$ &c.

Again, if the Logarithm of a decreasing *ratio* be sought, the infinite Root of $1-q$ or $\overline{1-q}|^{\frac{1}{m}}$ is $1 - \frac{1}{m}q - \frac{1}{2m}q^2 - \frac{1}{3m}q^3 - \frac{1}{4m}q^4 - \frac{1}{5m}q^5 - \frac{1}{6m}q^6$ &c. whence the decrement of the first of our infinite Number of Proportionals will be $\frac{1}{m}$ into $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$ &c. which therefore will be as the Logarithm of the *ratio* of Unity to $1-q$. But if m be put 10000 &c. then the said Logarithm will be $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$ &c.

Hence

Hence the terms of any *ratio* being a and b, q becomes $\frac{b-a}{a}$ or the difference divided by the leſſer term, when 'tis an increaſing *ratio*; or $\frac{b-a}{b}$ when 'tis decreaſing or as b to a. Whence the Logarithm of the ſame *ratio* may be doubly expreſt, for putting x for the difference of the terms a and b, it will be either

$$\frac{1}{m} \text{ into } \frac{x}{b} + \frac{x^2}{2bb} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} \text{ \&c. or}$$

$$\frac{1}{m} \text{ into } \frac{x}{a} - \frac{x^2}{2aa} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} \text{ \&c.}$$

But if the *ratio* of a to b be ſuppoſed divided into two parts, *viz.* into the *ratio* of a to the Arithmetical Mean between the terms, and the *ratio* of the ſaid Arithmetical Mean to the other term b, then will the Sum of the Logarithms of thoſe two *rationes* be the Logarithm of the *ratio* of a to b; and ſubſtituting $\frac{1}{2}z$ inſtead of $\frac{1}{2}a + \frac{1}{2}b$ the ſaid Arithmetical Mean, the Logarithms of thoſe *rationes* will be by the foregoing Rule,

$$\frac{1}{m} \text{ in } \frac{x}{z} + \frac{xx}{2zz} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} \text{ \&c. and}$$

$$\frac{1}{m} \text{ in } \frac{x}{z} - \frac{xx}{2zz} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} \text{ \&c.}$$

the Sum whereof $\frac{1}{m}$ in $\frac{2x}{z} * + \frac{2x^3}{3z^3} * + \frac{2x^5}{5z^5} * \frac{2x^7}{7z^7}$, &c. will be the Logarithm of the *ratio* of a to b, whoſe difference is x and Sum z. And this *Series* converges twice as ſwift as the former, and therefore is more proper for the Practice of making of Logarithms: Which it performs with that expedition, that where x the difference is but the hundredth part of the Sum, the firſt ſtep $\frac{2x}{z}$ ſuffices to ſeven places of the Logarithm, and the ſecond ſtep to twelve; But if *Briggs*'s firſt Twenty Chiliads of Logarithms be ſuppoſed made, as he has very carefully computed them, to fourteen places, the firſt ſtep alone is capable to give the Logarithm of any intermediate Number true to all the places of thoſe Tables.

After the ſame manner may the difference of the ſaid two Logarithms be very fitly applyed to find the Logarithms of Prime Numbers, having the Logarithms of the two next Numbers above and below them: For the difference of the *ratio* of a to $\frac{1}{2}z$ and of $\frac{1}{2}z$ to b is the *ratio* of ab to $\frac{1}{4}zz$, and the half of that *ratio* is that of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the Geometrical Mean to the Arithmetical. And conſequently the Logarithm thereof will be the half difference of the Logarithms of thoſe *rationes*, *viz.*

$$\frac{1}{m} \text{ into } \frac{xx}{2zz} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z_8} \text{ \&c.}$$

Which is a Theorem of good diſpatch to find the Logarithm of $\frac{1}{2}z$. But the ſame is yet much more advantageouſly performed by a Rule derived from the foregoing, and beyond which in my Opinion nothing better can be hoped. For the *ratio* of ab to $\frac{1}{4}zz$ or $\frac{1}{4}aa + \frac{1}{2}ab + \frac{1}{4}bb$, has the difference of its terms $\frac{1}{4}aa - \frac{1}{2}ab + \frac{1}{4}bb$ or the Square of $\frac{1}{2}a - \frac{1}{2}b = \frac{1}{4}xx$, which in the preſent caſe of finding the Logarithms of Prime Numbers is always Unity, and calling the Sum of the terms $\frac{1}{4}zz + ab = yy$, the Logarithm of the *ratio* of $\sqrt{ab}$ to $\frac{1}{2}a + \frac{1}{2}b$ or $\frac{1}{2}z$ will be found

$$\frac{1}{m} \text{ in } \frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} \text{ \&c.}$$

which converges very much faſter than any Theorem hitherto publiſhed for this purpoſe.

Here note that $\frac{1}{m}$ is all along applyed to adapt theſe Rules to all ſorts of Logarithms. If m be 10000 &c. it may be neglected, and you will have *Napeir*'s Logarithms, as was hinted before; but if you deſire *Briggs*'s Logarithms, which are now generally received, you muſt divide your Series by

2,302585092994045684017991454684364207601101488628772976033328

or multiply it by the reciprocal thereof, *viz.*

0,434294481903251827651128918916605082294397005803666566114454

But to ſave ſo operoſe a Multiplication (which is more than all the reſt of the Work) it is expedient to Divide this Multiplicator by the Powers of z or y continually, according to the direction of the Theorem, eſpecially where x is ſmall and Integer, reſerving the proper Quotes to be added together, when you have produced your Logarithm to as many

many Figures as you desire, of which Method I will give a Specimen.

If the Curiosity of any Gentleman that has leisure would prompt him to undertake to do the Logarithms of all Prime Numbers under 100000 to 25 or 30 Figures, I dare assure him that the facility of this Method will invite him thereto, nor can any thing more easie be desired. And to encourage him, I here give the Logarithms of the first Prime Numbers under 20 to sixty places, computed by the accurate Pen of Mr. *Abraham Sharp*, (from whose Industry and Capacity the World may in time expect great Performances) as they were communicated to me by our common Friend Mr. *Euclid Speidall*.

Numb.	Logarithm.
2	0,301029995663981195213738894724493026768189881462108541310427
3	0,477121254719662437295027903255115309200128864190695864829866
7	0,845098040014256830712216258592636193483572396323965406503835
11	1,041392685158225040750199971243024241706702190466453094596539
13	1,113943352306837769206541895026246254561189005053673288598083
17	1,230448921378273028540169894328337030007567378425046397380368
19	1,278753600952828961536333475756929317951129337394497598906819

The next Prime Number is 23, which I will take for an Example of the foregoing Doctrine; and by the first Rules, the Logarithm of the *ratio* of 22 to 23 will be found to be either

$$\frac{1}{22} - \frac{1}{968} + \frac{1}{31944} - \frac{1}{937024} + \frac{1}{25768160} \text{ \&c. or}$$

$$\frac{1}{23} + \frac{1}{1058} + \frac{1}{36501} + \frac{1}{1119364} + \frac{1}{32181715} \text{ \&c.}$$

As likewise that of the *ratio* of 23 to 24 by a like Process.

$$\frac{1}{23} - \frac{1}{1058} + \frac{1}{36501} - \frac{1}{1119364} + \frac{1}{32181715} \text{ \&c. or}$$

$$\frac{1}{24} + \frac{1}{1152} + \frac{1}{41472} + \frac{1}{1327104} + \frac{1}{39813120} \text{ \&c.}$$

And this is the Reſult of the Doctrine of *Mercator*, as improved by the Learned Dr. *Wallis*. But by the ſecond Theorem, *viz.* $\frac{2x}{z}+\frac{2x^3}{3z^3}+\frac{2x^5}{5z^5}$ &c. the ſame Logarithms are obtained by fewer ſteps. To wit,

$$\frac{2}{45}+\frac{2}{273375}+\frac{2}{922640625}+\frac{2}{26156861718 75}\ \text{\&c. and}$$

$$\frac{2}{47}+\frac{2}{311469}+\frac{2}{1146725035}+\frac{2}{3546361843241}\ \text{\&c.}$$

which was invented and demonſtrated in the Hyperbolick Spaces Analogous to the Logarithms, by the Excellent Mr. *James Gregory*, in his *Exercitationes Geometricæ*, and ſince further proſecuted by the aforeſaid Mr. *Speidall*, in a late Treatiſe in *Engliſh* by him publiſhed on this Subject. But the Demonſtration as I conceive was never till now perfected without the conſideration of the Hyperbola, which in a matter purely Arithmetical as this is, cannot ſo properly be applyed. But what follows I think I may more juſtly claim as my own, *viz.* That the Logarithm of the *ratio* of the Geometrical Mean to the Arithmetical between 22 and 24, or of $\sqrt{528}$ to 23 will be found to be either

$$\frac{1}{1058}+\frac{1}{1119364}+\frac{1}{888215334}+\frac{1}{626487882248}\ \text{\&c. or}$$

$$\frac{1}{1057}+\frac{1}{3542796579}+\frac{1}{659676558485285}\ \text{\&c.}$$

All theſe *Series* being to be multiplyed into 0,4342944819 &c. if you deſign to make the Logarithm of *Briggs*. But with great Advantage in reſpect of the Work, the ſaid 4342944819 &c. is divided by 1057, and the Quotient thereof again divided by three times the Square of 1057, and that Quotient again by $\frac{5}{3}$ of that Square, and that Quotient by $\frac{7}{5}$ thereof, and ſo forth, till you have as many Figures of your Logarithm as you deſire. As for Example, The Logarithm of the Geometrical Mean between 22 and 24 is found by the Logarithms of 2, 3 and 11 to be

	1.36131696126690612945009172669805
1057)43429 &c.(	41087462810146814347315886368
3 in 1117249)41087 &c.(	12258521544181829460074
$\frac{5}{3}$ in 1117249)12258 &c.(	6583235184376175
$\frac{7}{5}$ in 1117249)65832 &c.(	4208829765
$\frac{9}{7}$ in 1117249)42088 &c.(	2930
Summa	1.36172783601759287886777711225117

Which is the Logarithm of 23 to thirty two places, and obtained by five Divisions with very small *Divisors*, all which is much less work than simply multiplying the *Series* into the said Multiplicator 43429 &c.

Before I pass on to the converse of this Problem, or to shew how to find the Number appertaining to a Logarithm assigned, it will be requisite to advertise the Reader, that there is a small mistake in the aforesaid Mr. *James Gregory's Vera Quadratura Circuli & Hyperbolæ*, published at *Padua Anno* 1667, wherein he applies his Quadrature of the Hyperbola to the making the Logarithms; In *pag*. 48. he gives the Computation of the Lord *Napeir's* Logarithm of 10, to five and twenty places, and finds it 2302585092994045624017870 instead of 2302585092994045684017991, erring in the eighteenth Figure, as I was assured upon my own Examination of the Number I here give you, and by comparison thereof with the same wrought by another hand, agreeing therewith to 57 of the 60 places. Being desirous to be satisfied how this difference arose, I took the no small trouble of examining Mr. *Gregory's* Work, and at length found that in the inscribed *Polygon* of 512 Sides, in the eighteenth Figure was a 0 instead of 9, which being rectified, and the subsequent Work corrected therefrom, the result did agree to a Unite with our Number. And this I propose not to Cavil at an easie mistake in managing of so vast Numbers, especially by a Hand that has so well deserved of the Mathematical Sciences, but to shew the exact coincidence of two so very differing Methods to make Logarithms, which might otherwise have been questioned.

From the Logarithm given to find what *ratio* it expresses, is a Problem that has not been so much considered as the former, but which is solved with the like ease, and demonstrated by a like Process, from the same general Theorem of Mr. *Newton*: For as the Logarithm of the *ratio* of 1 to $1+q$ was proved to be $\overline{1+q}|^{\frac{1}{m}}-1$, and that of the *ratio* of 1 to $1-q$ to be $1-\overline{1-q}|^{\frac{1}{m}}$: so the Logarithm, which we will from henceforth call L, being given, $1+L$ will be equal to $\overline{1+q}|^{\frac{1}{m}}$ in the one case; and $1-L$ will be equal to

$1-q$

$\overline{1-q}|^{\frac{1}{m}}$ in the other: Consequently $\overline{1+L}|^{m}$ will be equal to $1+q$, and $\overline{1-L}|^{m}$ to $1-q$; that is, according to Mr. *Newton's* said Rule, $1+mL+\frac{1}{2}m^2L^2+\frac{1}{6}m^3L^3+\frac{1}{24}m^4L^4+\frac{1}{120}m^5L^5$ &c. will be $=1+q$, and $1-mL+\frac{1}{2}m^2L^2-\frac{1}{6}m^3L^3+\frac{1}{24}m^4L^4-\frac{1}{120}m^5L^5$ &c. will be equal to $1-q$, m being any infinite Index whatsoever, which is a full and general Proposition from the Logarithm given to find the Number, be the *Species* of Logarithm what it will. But if *Napeir's* Logarithm be given, the Multiplication by m is saved, (which Multiplication is indeed no other than the reducing the other *Species* to his) and the *Series* will be more simple, *viz.* $1+L+\frac{1}{2}LL+\frac{1}{6}L^3+\frac{1}{24}L^4+\frac{1}{120}L^5$ &c. or $1-L+\frac{1}{2}LL-\frac{1}{6}L^3+\frac{1}{24}L^4-\frac{1}{120}L^5$ &c. This *Series*, especially in great Numbers, converges so slowly, that it were to be wished it could be contracted.

If one term of the *ratio*, whereof L is the Logarithm, be given, the other term will be had easily by the same Rule: For if L were *Napeir's* Logarithm of the *ratio* of a the lesser to b the greater term, b would be the Product of a into $1+L+\frac{1}{2}LL+\frac{1}{6}LLL$ &c. $=a+aL+\frac{1}{2}aLL+\frac{1}{6}aL^3$ &c. But if b were given, a would be $=b-bL+\frac{1}{2}bLL-\frac{1}{6}bL^3$ &c. Whence by the help of the *Chiliads*, the Number appertaining to any Logarithm will be exactly had to the utmost extent of the Tables. If you seek the nearest next Logarithm, whether greater or lesser, and call its Number a if lesser, or b if greater than the given L, and the difference thereof from the said nearest Logarithm you call l; it will follow that the Number answering to the Logarithm L will be either a into $1+l+\frac{1}{2}ll+\frac{1}{6}lll+\frac{1}{24}l^4+\frac{1}{120}l^5$ &c. or else b into $1-l+\frac{1}{2}ll-\frac{1}{6}lll+\frac{1}{24}l^4-\frac{1}{120}l^5$ &c. wherein as l is less, the *Series* will converge the swifter. And if the first 20000 Logarithms be given to fourteen places, there is rarely occasion for the three first steps of this *Series* to find the Number to as many places. But for *Ulacqs* great Canon of 100000 Logarithms, which is made but to ten places, there is scarce ever need for more than the first step $a+al$ or $a+mal$ in one case, or else $b-bl$ or $b-mbl$ in the other, to have the Number true to as many Figures as those Logarithms consist of.

If future Induſtry ſhall ever produce Logarithmick Tables to many more places, than now we have them; the aforeſaid Theorems will be of more uſe to deduce the correſpondent Natural Numbers to all the places thereof. In order to make the firſt *Chiliad* ſerve all Uſes, I was deſirous to contract this *Series*, wherein all the Powers of l are preſent, into one, wherein each alternate Power might be wanting; but found it neither ſo ſimple or uniform as the other. Yet the firſt ſtep thereof is I conceive moſt commodious for Practice, and withal exact enough for Numbers not exceeding fourteen places, ſuch as are Mr. *Briggs*'s large Table of Logarithms; and therefore I recommend it to common Uſe. It is thus: $a + \frac{al}{1 - \frac{1}{2}l}$ or $b - \frac{bl}{1 + \frac{1}{2}l}$ will be the Number anſwering to the Logarithm given, differing from the truth by but one half of the third ſtep of the former *Series*. But that which renders it yet more eligible is, that with equal facility it ſerves for *Briggs*'s or any other ſort of Logarithms, with the only variation of writing $\frac{1}{m}$ inſtead of 1, that is, $a + \frac{al}{\frac{1}{m} - \frac{1}{2}l}$ and $b - \frac{bl}{\frac{1}{m} + \frac{1}{2}l}$, or $\frac{\frac{1}{m}a + \frac{1}{2}la}{\frac{1}{m} - \frac{1}{2}l}$ and $\frac{\frac{1}{m}b - \frac{1}{2}lb}{\frac{1}{m} + \frac{1}{2}l}$, which are eaſily reſolved into Analogies, *viz.*

As 43429 &c. $- \frac{1}{2}l$ to 43429 $+ \frac{1}{2}l$: : So is a } to the Number ſought.
or As 43429 &c. $+ \frac{1}{2}l$ to 43429 $- \frac{1}{2}l$: : So is b }

If more ſteps of this *Series* be deſired, it will be found as follows, $a + \frac{al}{1 - \frac{1}{2}l} - \frac{\frac{1}{12}al^3}{1 - l} + \frac{\frac{1}{35}al^5}{1 - 2l}$ &c. as may eaſily be demonſtrated by working out the Diviſions in each ſtep, and collecting the Quotes, whoſe Sum will be found to agree with our former *Series*.

Thus I hope I have cleared up the Doctrine of Logarithms, and ſhewn their Conſtruction and Uſe independent from the *Hyperbola*, whoſe Affections have hitherto been made uſe of for this purpoſe, though this be a matter purely Arithmetical, nor properly demonſtrable from the Principles of Geometry. Nor have I been obliged to have recourſe to the Method of Indiviſibles, or the Arithmetick of Infinites, the whole being no other than an eaſie Corollary to Mr. *Newton*'s General Theorem for forming Roots and Powers.

V. A